**Fibonacci Numbers and Some Explorations**

In mathematics, the Fibonacci numbers or the Fibonacci sequence are the numbers in following manner/sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, … … …

Or

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, … … …

By definition, the first two numbers in the Fibonacci sequence are 0 and 1, or 1 and 1, depending on the chosen starting point of the sequence, and each subsequent number is the sum of the previous two.

In mathematics, the sequence $F\_{n}$of Fibonacci numbers is defined by the recurrence relation

$$F\_{n}=F\_{n-1}+F\_{n-2}$$

With seed values $F\_{1}=1, F\_{2}=1$ or $F\_{0}=0, F\_{1}=1.$

We can use Excel to generate Fibonacci numbers. The first 50 numbers of the Fibonacci sequence using Excel is [here](Fibonacci%20Numbers.xlsx).

The Fibonacci sequence exhibits a certain numerical pattern which originated as the answer to an exercise in the first ever high school algebra text. This pattern turned out to have an interest and importance far beyond what its creator imagined. It can be used to model or describe an amazing variety of phenomena, in mathematics, science, art, and nature. The mathematical ideas of the Fibonacci sequence leads to, such as the golden ratio, spirals and self- similar curves, have long been appreciated for their charm and beauty, but no one can really explain why they are echoed so clearly in the world of art and nature (Beck & Ross, 2010).

The story began in Pisa, Italy in the year 1202. Leonardo Pisano Bigollo was a young man in his twenties, a member of an important trading family of Pisa. In his travels throughout the Middle East, he was captivated by the mathematical ideas that had come to the West from India through the Arabic countries. When he returned to Pisa he published these ideas in a book on mathematics called *Liber Abaci*, which became a landmark in Europe. Leonardo, who has since came to be known as Fibonacci, became the most celebrated mathematician of the middle Ages. His book was a discourse on mathematical methods in commerce. However, the book is now remembered for the attention he brought to the Europe of the Hindu system for writing numbers. European tradesmen and scholars were still clinging to the use of the old Roman numerals; modern mathematics would have been impossible without this change to the Hindu system, which we call now Arabic notation, since it came west through Arabic lands (Parmanand, 1985).

Below are some interesting use/cases of Fibonacci Numbers.

Fibonacci numbers and Pascal’s Triangle

Is there any relationship between Fibonacci Numbers and Pascal’s Triangle?

Let’s Explore.

A Pascal’s Triangle looks like this:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

… … … … … … … … …

where each entry in the triangle is the sum of the two numbers above it.

We can also realign the triangle to look like the table below:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **0** | **1** | **2** | **3** | **4** | **...** |
| 1 |  | **0** | 1 |  |  |  |  |  |
| 1 1 | **1** | 1 | 1 |  |  |  |  |
| 1 2 1 | **2** | 1 | 2 | 1 |  |  |  |
| 1 3 3 1 | **3** | 1 | 3 | 3 | 1 |  |  |
| 1 4 6 4 1 | **4** | 1 | 4 | 6 | 4 | 1 |  |
| ... | **...** | ... |  |  |  |  |  |

So here, each number on the table is the sum of the number above and the number to the left of that one. Each blank space is counted as 0. For example, on the last row, 6 is the sum of the 3 and 3. Also, note that each row starts and ends with 1.

If we look carefully, we can find Fibonacci Numbers in Pascal’s Triangle. Let’s look at the diagram below:

**Fibonacci Numbers**

|  |  |  |
| --- | --- | --- |
|  |  | 1 |
| **1** | 1 | 1The **green diagonal** sums to 8;the **blue diagonal** sums to 13;the **purple diagonal** sums to 21. |
| **1** | 1 | 2 | 1 |
| **2** | 1 | 3 | 3 | 1 |
| **3** | 1 | 4 | 6 | 4 | 1 |
| **5** | 1 | 5 | 10 | 10 | 5 | 1 |
| **8** | 1 | 6 | 15 | 20 | 15 | 6 | 1 |
| **13** | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |
| **21** | ... |  |  |  |  |  |  |  |

Each **purple number** is the sum of a **blue** and a **green** number on the row above.

Because all the numbers in Pascal's Triangle are made the same way - by adding the two numbers *above and to the left on the row above*, then we can see that each **purple number** is just the sum of a **green number** and a **blue number** and we use up all the **blue** and **green** numbers to make all the **purple** ones.

The sum of all the purple numbers is therefore the same as the sum of all the blues and all the greens: 8+13=21

The general principle that we have just illustrated is:

The *sum* of the numbers on one diagonal is the sum of the numbers on the previous two diagonals. If we let $D(i)$ be the sum of the numbers on the Diagonal that starts with one of the extra zeros at the beginning of row $i$, then

$$D\left(0\right)= 0 and D\left(1\right)=1$$

are the two initial diagonals shown in the table above. The green diagonal sum is $D(6)=8$ (since its extra initial zero is in row 6) and the blue diagonal sum is $D(7)$ which is 13. Our purple diagonal is $D\left(8\right)=21=D\left(6\right)+D\left(7\right).$

We also have shown that this is always true: one diagonals sum is the sum of the previous two diagonal sums, or, in terms of our D series of numbers:

$$D\left(i\right)=D\left(i-1\right)+D(i-2)$$

But,

$$D\left(0\right)=1$$

$$D\left(1\right)=1$$

$$D\left(i\right)=D\left(i-1\right)+D(i-2)$$

is exactly the definition of the Fibonacci numbers! So $D(i)$ is just F$(i)$ and

*the sums of the diagonals in Pascal's Triangle are the Fibonacci numbers!*

Fibonacci Spiral diagram

We can make a diagram with the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, ……… $. If we start with two small squares of size 1 next to each other. On top of both of these draw a square of size 2 (=1+1).
We can now draw a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long; and then another touching both the 2-square and the 3-square (which has sides of 5 units). We can continue adding squares around the diagram, *each new square having a side as long as the sum of the latest two square's sides*. This set of rectangles whose sides are two successive Fibonacci numbers in length and which are composed of squares with sides which are Fibonacci numbers; it is called the **Fibonacci Rectangles.**

Wherever we stop, we will always get a rectangle, since the next square to add is determined by the longest edge on the current rectangle. Also, those longest edges are just the sum of the latest two sides-of-squares to be added. Since we start with squares of sides 1 and 1, this tells us why the squares sides are the Fibonacci numbers (the next is the sum of the previous 2).

Now, let’s express each rectangle's area as a sum of its component square areas:

The diagram shows that

$$1^{2}+1^{2}+2^{2}+3^{2}+5^{2}+8^{2}+13^{2}=13×7×3=13×21$$

and also, the smaller rectangles show:

$$1^{2}+1^{2}=2=1×2$$

$$1^{2}+1^{2}+2^{2}=2×3$$

$$1^{2}+1^{2}+2^{2}+3^{2}=3×5$$

$$1^{2}+1^{2}+2^{2}+3^{2}+5^{2}=5×8$$

$$1^{2}+1^{2}+2^{2}+3^{2}+5^{2}+8^{2}=8×13$$

This actually shows that the pattern will work for any number of squares of Fibonacci numbers that we wish to sum. They always total to the largest Fibonacci number used in the squares multiplied by the next Fibonacci number.

Now, to express this as a mathematical expression:

$1^{2}+1^{2}+2^{2}+3^{2}+5^{2}………+F\left(n\right)^{2}=F\left(n\right)F(n+1)$*, and it is true for any n from 1 upwards.*

An extension of Fibonacci Rectangles is to create a spiral. Follow this Khan Academy video on “[Doodling in Math: Spirals, Fibonacci, and being a plant](https://www.khanacademy.org/math/recreational-math/vi-hart/spirals-fibonacci/v/doodling-in-math--spirals--fibonacci--and-being-a-plant--1-of-3)”.

A Fibonacci Number Trick

Here is a trick we used to perform on friends to show them I have amazing mathematical power!!! This was during my school days. Back then I only knew it works but never investigated why it works. While browsing on internet, I came across an elegant way to prove it as well (Long, 1985). First I will present the trick then show the proof.

**The Steps:**

I) Choose any two numbers (and of course don’t reveal any number). Try to keep the numbers small as you have to do some adding yourself. Write them as if you are going to add them up. *For example, let’s choose 15 and 17.*

II) Now add the two numbers and write the sum underneath the second number to make the third entry in the column. *For example:* $15$

$17$

 *32*

III) Now add the second and third numbers and again write their sum underneath to make the fourth entry in the column: *For example:* $15$

$17$

$32$

 *49*

|  |
| --- |
| 15 |
| 17 |
| 32 |
| 49 |
| 81 |
| 130 |
| 211 |
| 341 |
| 552 |
| 893 |

IV) Keep on doing this, adding the number you have just written to the number before it and putting the new sum underneath. Stop when you have 10 numbers written down and draw a line under the tenth. *For example, the list should look like this:*

Now, once you reveal me your 10 numbers, I can give you the sum of those numbers immediately!!! *For example: I can tell the sum 2321 in couple of seconds.*

**What is the Trick?**

For me, to be able to tell you the sum of 10 numbers above, I just need to look at the 4th number from bottom and times it by 11!!!

**Why it Works?**

Turns out, the trick is in Fibonacci sequence!!! Let’s use algebraic expression for the numbers picked. Suppose the first two numbers are $x$ and$ y$. So the sequence develops as follows:

|  |  |
| --- | --- |
| 1st Number | x |
| 2nd Number | y |
| 3rd Number | x+y |
| 4th Number | x+2y |
| 5th Number | 2x+3y |
| 6th Number | 3x+5y |
| 7th Number | 5x+8y |
| 8th Number | 8x+13y |
| 9th Number | 13x+21y |
| 10th Number | 21x+34y |
| **Total** | 55x+88y |

Now look at the fourth number from the bottom. That is 5x+8y, and the sum is 11 times the number: $11\left(5x+8y\right)=55x+88y$.

So the trick works by a special property of adding up exactly ten numbers from a Fibonacci-like sequence and will work for any two starting values $x$ and $y$!!! *Also, did you notice that the multiples of* $x$ *and* $y$ *were the Fibonacci numbers?*

# Bibliography

Beck, M., & Ross, G. (2010). *The Art of Proof: Basic Training for Deeper Mathematics.* New York: Springer.

Long, C. T. (1985). On a Fibonacci Arithmatical Trick. *Fibonacci Quarterly* *, 23*, 221-231.

Parmanand, S. (1985). The So-called Fibonacci numbers in ancient and medieval India. *Historia Mathematics* *, 12* (3), 229-244.